ELSEVIER

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Journal of Computational and Applied Mathematics 53 (1994) 185–206

Mass- and momentum-conserving spectral methods for Stokes flow

Robert G. Owens [a,1,2], Timothy N. Phillips [b,*]

[a] *Department of Mathematics, Boğaziçi University, 80815 Bebek, Istanbul, Turkey*

[b] *Department of Mathematics, The University of Wales, Aberystwyth, Dyfed SY23 3BZ, United Kingdom*

Received 19 May 1992; revised 10 November 1992

Abstract

The governing equations for Stokes flow are formulated in terms of a stream function and Airy stress function. This formulation ensures that mass and momentum are conserved identically. In terms of these new variables, the equations of motion are written as a second-order elliptic system. These equations are embedded in biharmonic equations and the boundary conditions appropriate for this higher-order system are determined using a least-squares process. This technique is applied to the planar stick-slip problem. A numerical solution to the problem is obtained using a spectral domain decomposition method. An algebraic mapping is used to treat the flow domain without truncation. The coefficients in a singular expansion of the stream function about the stick-slip singularity are computed using a post-processing technique.

Keywords: Airy stress function; Stream function; Spectral methods; Stick-slip problem; Elliptic system

Dedicated to Professor Ken Walters FRS on the occasion of his sixtieth birthday

1. Introduction

A formulation of the Stokes problem in which mass and momentum are conserved identically is considered. This is facilitated by introducing a stream function and Airy stress function, respectively. The constitutive equation can then be written in terms of these new variables. This results in a coupled second-order elliptic system of partial differential equations. A least-squares formulation of this system yields biharmonic equations for the stream function and the Airy stress function. This

* Corresponding author.

[1] The work of this author was partly supported by a research studentship funded by the United Kingdom Science and Engineering Research Council while he was in residence at The University of Wales, Aberystwyth.

[2] Present address: Department of Mathematics, Napier University, 219 Colinton Road, Edinburgh EH14 1DJ, United Kingdom.

SSDI 0377-0427(92)00136-3

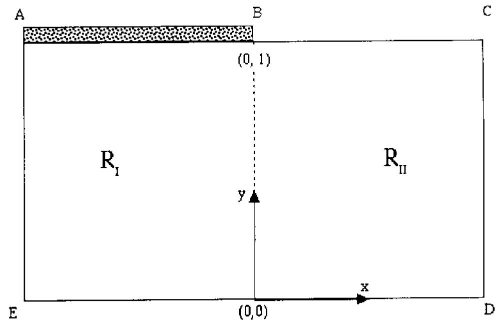

Fig. 1. The stick-slip geometry and its domain decomposition.

process also enables the appropriate boundary conditions needed to solve the higher-order system to be determined. The additional boundary conditions are shown to be sufficient to ensure that the solution of the higher-order system is in fact a solution of the second-order system.

The Stokes problem for the planar stick-slip geometry is considered in this paper. The geometry is shown schematically in Fig. 1. The problem was first considered by Richardson [20] who obtained a closed-form solution using the Wiener–Hopf method. Sturges [21] solved the same problem by expanding the solution in the slip region as a Fourier series, the solution in the walled channel as a biorthogonal series and matching the velocities and stresses across the common interface. Trogdon and Joseph [22] solved the Stokesian stick-slip problem for a round jet using both the Wiener–Hopf method of [20] and the eigenfunction method of [21]. Coleman [5] used a contour integral formulation to solve the problem. Georgiou et al. [8] developed a singular finite-element method to take account of the boundary singularity at the point $(0, 1)$ and obtained more accurate results than those obtained using ordinary finite-element meshes.

A spectral domain decomposition method for determining the stream function and the Airy stress function is described. The flow domain is divided into two semi-infinite subdomains. These subdomains are mapped onto finite rectangles using an algebraic mapping technique which enables the flow domain to be treated without truncation. These techniques have been used for Newtonian flow through a planar contraction geometry [12,13]. However, previous work [12,13] using spectral methods has determined the stream function only. In the present study the pressure and the components of stress are also calculated. We note that the components of stress may also be calculated from a knowledge of the stream function, but these will not, in general, satisfy the momentum equation.

2. The governing equations

The governing equations in fluid mechanics consist of field equations and constitutive equations. The field equations comprise the equation of continuity and the conservation of momentum. For the inertialess planar flow of an incompressible fluid these statements assume the mathematical form

$$\nabla \cdot \boldsymbol{v} = 0, \tag{1}$$

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}, \tag{2}$$

where $\boldsymbol{v}$ denotes the velocity vector and $\boldsymbol{\sigma}$ the Cauchy stress tensor. The constitutive equation for a Newtonian fluid, suitably nondimensionalized, is given by

$$T = 2d, \tag{3}$$

where T is the extra-stress tensor and d is the rate-of-deformation tensor. The Cauchy stress and extra-stress are related to the pressure by

$$\sigma = -pI + T. \tag{4}$$

The velocity components may be expressed in terms of a stream function $\psi(x, y)$ by

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}, \tag{5}$$

which ensures that (1) is satisfied. Suppose now that the components of the Cauchy stress tensor are given in terms of the Airy stress function $\phi(x, y)$ by

$$\sigma_{xx} = -p + T_{xx} = \frac{\partial^2\phi}{\partial y^2}, \qquad \sigma_{xy} = T_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y}, \qquad \sigma_{yy} = -p + T_{yy} = \frac{\partial^2\phi}{\partial x^2}; \tag{6}$$

then the momentum equation (2) is satisfied identically. Substitution of (5) and (6) into (3) results in the following system of equations for ψ, ϕ and p:

$$L\phi = 2M\psi, \tag{7}$$

$$M\phi = -2L\psi, \tag{8}$$

$$\Delta\phi = -2p, \tag{9}$$

where Δ denotes the Laplacian operator and L and M are the hyperbolic operators given by

$$L = \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}, \qquad M = 2\frac{\partial^2}{\partial x\,\partial y}.$$

Note that the pressure does not appear explicitly in (7) and (8). However it may be found from (9), if required. The use of the identity

$$\Delta^2 = L^2 + M^2$$

enables (7) and (8) to be simplified further to give

$$\Delta^2\phi = 0, \qquad \Delta^2\psi = 0. \tag{10}$$

Clearly, a complete description of the velocity field and Cauchy stress tensor may be obtained by solving (10), subject to boundary conditions appropriate to the flow configuration under consideration. It is the purpose of the present paper to establish natural boundary conditions for the Stokes problem using variational techniques. Furthermore, these boundary conditions are shown to be sufficient to ensure that the solution to (10) also satisfies the second-order elliptic system (7), (8).

In the stick-slip geometry the channel has a no-slip surface $y = 1$ for $x \leqslant 0$ and a shear free (slip) surface $y = 1$ for $x > 0$. The centreline $y = 0$ is a line of symmetry. In terms of the stream function these conditions become

$$\psi(x, 1) = 1, \quad \frac{\partial\psi}{\partial y}(x, 1) = 0, \qquad x \leqslant 0, \tag{11}$$

$$\psi(x,1) = 1, \quad \frac{\partial^2\psi}{\partial y^2}(x,1) = 0, \qquad x > 0, \tag{12}$$

$$\psi(x,0) = 0, \quad \frac{\partial^2\psi}{\partial y^2}(x,0) = 0, \qquad -\infty < x < \infty. \tag{13}$$

Suppose that the flow is driven by a constant pressure gradient P, say, at entry; then,

$$\frac{\partial p}{\partial x}(-\infty, y) = P.$$

If we assume that the flow is fully developed so that the components of the extra-stress tensor are functions of y only, then from (2) and (3) we are able to derive that

$$P = \frac{\partial^3\psi}{\partial y^3}(-\infty, y).$$

We solve this differential equation for ψ subject to conditions (11) and (13) and find

$$P = -3, \qquad \psi(-\infty, y) = \tfrac{1}{2}y(3 - y^2). \tag{14}$$

Similarly, if Q is the constant pressure gradient at exit, then

$$Q = 0, \qquad \psi(\infty, y) = y. \tag{15}$$

3. Least-squares formulation

In this section we derive the biharmonic equations (10) for ψ and ϕ from a least-squares formulation of the elliptic system (7), (8). An important outcome of this procedure is the derivation of natural boundary conditions necessary to solve (10) for ϕ as well as ψ.

Theorem 3.1. *Let q, r, u and v be functions which are four times continuously differentiable in a simply connected domain Ω bounded by a simple closed curve Γ. Then,*

$$\begin{aligned}
&\int\!\!\int_\Omega \{(Lq - 2Mr)(Lu - 2Mv) + (Mq + 2Lr)(Mu + 2Lv)\}\,\mathrm{d}S \\
&\quad = -\int_\Gamma \frac{\partial u}{\partial \nu}(Lq - 2Mr)\,\mathrm{d}s - \int_\Gamma u\left\{2\frac{\partial}{\partial \tau}(Mq + 2Lr) - \frac{\partial}{\partial \nu}(Lq - 2Mr)\right\}\mathrm{d}s \\
&\qquad + \int\!\!\int_\Omega u\,\Delta^2 q\,\mathrm{d}S - 2\int_\Gamma \frac{\partial v}{\partial \nu}(Mq + 2Lr)\,\mathrm{d}s \\
&\qquad + 2\int_\Gamma v\left\{\frac{\partial}{\partial \nu}(Mq + 2Lr) + 2\frac{\partial}{\partial \tau}(Lq - 2Mr)\right\}\mathrm{d}s \\
&\qquad + 4\int\!\!\int_\Omega v\,\Delta^2 r\,\mathrm{d}S,
\end{aligned} \tag{16}$$

where $\partial/\partial\nu$ and $\partial/\partial\tau$ represent differentiation in the directions $\boldsymbol{\nu} = (\mathrm{d}y/\mathrm{d}s, \mathrm{d}x/\mathrm{d}s)$ and $\boldsymbol{\tau} = (-\mathrm{d}x/\mathrm{d}s, \mathrm{d}y/\mathrm{d}s)$, respectively.

Proof. Let us define F and G in terms of q and r by

$$F = Lq - 2Mr, \qquad G = Mq + 2Lr. \tag{17}$$

The left-hand side I of (16) may be written in the equivalent form

$$\begin{aligned}
I &= \int\int_{\Omega} \left\{ -F \frac{\partial^2 u}{\partial x^2} + 2G \frac{\partial^2 u}{\partial x \partial y} + F \frac{\partial^2 u}{\partial y^2} - 2G \frac{\partial^2 v}{\partial x^2} - 4F \frac{\partial^2 v}{\partial x \partial y} + 2G \frac{\partial^2 v}{\partial y^2} \right\} \mathrm{d}S \\
&= \int\int_{\Omega} \left\{ \left[-F \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + G \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \right] + \left[G \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) + F \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \right] \right. \\
&\qquad \left. + \left[-2G \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) - 2F \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) \right] + \left[-2F \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + 2G \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) \right] \right\} \mathrm{d}S.
\end{aligned} \tag{18}$$

We apply Green's theorem in the plane to each of the four terms in the above integral to give

$$\begin{aligned}
I &= \int_{\Gamma} \left\{ \frac{\partial u}{\partial x}(-F\,\mathrm{d}y - G\,\mathrm{d}x) + \frac{\partial u}{\partial y}(G\,\mathrm{d}y - F\,\mathrm{d}x) \right. \\
&\qquad \left. + \frac{\partial v}{\partial x}(-2G\,\mathrm{d}y + 2F\,\mathrm{d}x) + \frac{\partial v}{\partial y}(-2F\,\mathrm{d}y - 2G\,\mathrm{d}x) \right\} \\
&\quad - \int\int_{\Omega} \left\{ \frac{\partial u}{\partial x} \left[-\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right] + \frac{\partial u}{\partial y} \left[\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y} \right] \right\} \mathrm{d}S \\
&\quad + 2 \int\int_{\Omega} \left\{ \frac{\partial v}{\partial x} \left[\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y} \right] + \frac{\partial v}{\partial y} \left[\frac{\partial F}{\partial x} - \frac{\partial G}{\partial y} \right] \right\} \mathrm{d}S.
\end{aligned}$$

Integration by parts and a second application of Green's theorem in the plane results in

$$\begin{aligned}
I &= -\int_{\Gamma} F \frac{\partial u}{\partial \nu}\,\mathrm{d}s - \int_{\Gamma} u \left\{ \left[-\frac{\partial F}{\partial x} + 2\frac{\partial G}{\partial y} \right] \mathrm{d}y - \left[2\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y} \right] \mathrm{d}x \right\} \\
&\quad + \int\int_{\Omega} u \left\{ -\frac{\partial^2 F}{\partial x^2} + 2\frac{\partial^2 G}{\partial x \partial y} + \frac{\partial^2 F}{\partial y^2} \right\} \mathrm{d}S - 2\int_{\Gamma} G \frac{\partial v}{\partial \nu}\,\mathrm{d}s \\
&\quad + 2\int_{\Gamma} v \left\{ \left[\frac{\partial G}{\partial x} + 2\frac{\partial F}{\partial y} \right] \mathrm{d}y - \left[2\frac{\partial F}{\partial x} - \frac{\partial G}{\partial y} \right] \mathrm{d}x \right\} \\
&\quad - 2\int\int_{\Omega} v \left\{ \frac{\partial^2 G}{\partial x^2} + 2\frac{\partial^2 F}{\partial x \partial y} - \frac{\partial^2 G}{\partial y^2} \right\} \mathrm{d}S.
\end{aligned}$$

We note that

$$-\frac{\partial^2 F}{\partial x^2} + 2\frac{\partial^2 G}{\partial x \partial y} + \frac{\partial^2 F}{\partial y^2} = \Delta^2 q \quad \text{and} \quad \frac{\partial^2 G}{\partial x^2} + 2\frac{\partial^2 F}{\partial x \partial y} - \frac{\partial^2 G}{\partial y^2} = -2\,\Delta^2 r.$$

Therefore we may recast I in the form

$$\begin{aligned}
I &= -\int_{\Gamma} F \frac{\partial u}{\partial \nu}\,\mathrm{d}s - \int_{\Gamma} u \left\{ 2\frac{\partial G}{\partial \tau} - \frac{\partial F}{\partial \nu} \right\} \mathrm{d}s + \int\int_{\Omega} u\,\Delta^2 q\,\mathrm{d}S \\
&\quad - 2\int_{\Gamma} G \frac{\partial v}{\partial \nu}\,\mathrm{d}s + 2\int_{\Gamma} v \left\{ \frac{\partial G}{\partial \nu} + 2\frac{\partial F}{\partial \tau} \right\} \mathrm{d}s + 4\int\int_{\Omega} v\,\Delta^2 r\,\mathrm{d}S.
\end{aligned} \tag{19}$$

If we substitute for F and G in (19) using the relations (17), then the result follows. □

Let Ω_h denote the flow domain for the stick-slip problem truncated at some finite distance h upstream and downstream of the stick-slip point $(0,1)$. The domain Ω_ϵ is a semi-circular region of radius $\epsilon \ll 1$ centred at the stick-slip point and lying within Ω_h. Thus,

$$\Omega_h = \{(x,y):\ x \in [-h,h],\, y \in [0,1]\}$$

and

$$\Omega_\epsilon = \{(x,y):\ x \in (-\epsilon,\epsilon),\, y \in (1-\sqrt{\epsilon^2-x^2}, 1]\}.$$

Let Γ_ϵ denote the semi-circular part of the boundary of Ω_ϵ and B^- and B^+ the points $(-\epsilon,1)$, $(\epsilon,1)$, respectively.

Consider the following minimization problem over the domain $\Omega = \Omega_h \setminus \Omega_\epsilon$. Find functions q and r which minimize $\mathcal{J}(q,r)$ over those functions belonging to some admissible class where

$$\mathcal{J}(q,r) = \int\int_\Omega \{(Lq-2Mr)^2 + (Mq+2Lr)^2\}\,\mathrm{d}S. \tag{20}$$

Suppose that $\mathcal{J}(q,r)$ attains its minimum value when $q=\phi$, $r=\psi$. Let us vary both ϕ and ψ keeping x and y fixed by considering the functions $\phi+\mu\eta$ and $\psi+\mu\zeta$, where μ is a real scalar. Then for ϕ and ψ to be stationary functions of the integral functional $\mathcal{J}$, we require (see [10])

$$\delta\mathcal{J} = 0,$$

where

$$\delta\mathcal{J} = 2\mu\int\int_\Omega \{(L\phi-2M\psi)(L\eta-2M\zeta) + (M\phi+2L\psi)(M\eta+2L\zeta)\}\,\mathrm{d}S. \tag{21}$$

We can apply the result of Theorem 3.1 to the integral expression in (21) to obtain

$$\begin{aligned}\delta\mathcal{J} = &-2\mu\int_\Gamma F\frac{\partial\eta}{\partial n}\,\mathrm{d}s - 2\mu\int_\Gamma \eta\left\{2\frac{\partial G}{\partial t} - \frac{\partial F}{\partial n}\right\}\mathrm{d}s + 2\mu\int\int_\Omega \eta\,\Delta^2\phi\,\mathrm{d}S\\ &-4\mu\int_\Gamma G\frac{\partial\zeta}{\partial n}\,\mathrm{d}s + 4\mu\int_\Gamma \zeta\left\{\frac{\partial G}{\partial n} + 2\frac{\partial F}{\partial t}\right\}\mathrm{d}s + 8\mu\int\int_\Omega \zeta\,\Delta^2\psi\,\mathrm{d}S,\end{aligned} \tag{22}$$

where now $F = L\phi - 2M\psi$ and $G = M\phi + 2L\psi$. Since the stream function is prescribed everywhere on the boundary Γ of Ω except on Γ_ϵ, the variation $\zeta = 0$ on $\Gamma\setminus\Gamma_\epsilon$ so that

$$\int_{\Gamma\setminus\Gamma_\epsilon} \zeta\left\{\frac{\partial G}{\partial n} + 2\frac{\partial F}{\partial t}\right\}\mathrm{d}s = 0. \tag{23}$$

From (22) we deduce that the functions ϕ and ψ that minimize $\mathcal{J}(q,r)$ must satisfy

$$\Delta^2\phi = 0, \quad \Delta^2\psi = 0, \quad \text{in } \Omega, \tag{24}$$

$$\frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial x^2} = 0, \quad \text{on } AE \text{ and } CD, \tag{25}$$

$$\frac{\partial^3\phi}{\partial x^3} + 3\frac{\partial^3\phi}{\partial x\,\partial y^2} = -4\frac{\partial^3\psi}{\partial y^3} = \begin{cases} 12, & \text{on } AE, \\ 0, & \text{on } CD, \end{cases} \tag{26}$$

$$\frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial x^2} = 0, \quad \frac{\partial^3\phi}{\partial y^3} + 3\frac{\partial^3\phi}{\partial x^2\,\partial y} = 0, \qquad \text{on } AB^-, \tag{27}$$

$$\frac{\partial^3\phi}{\partial y^3} + 3\frac{\partial^3\phi}{\partial x^2\,\partial y} = 0, \quad \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial x^2} - 4\frac{\partial^2\psi}{\partial x\,\partial y} = 0, \qquad \text{on } B^+C \text{ and } DE, \tag{28}$$

$$\frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial x^2} - 4\frac{\partial^2\psi}{\partial x\,\partial y} = 0, \quad \frac{\partial^2\phi}{\partial x\,\partial y} + \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2} = 0, \qquad \text{on } \Gamma_\epsilon, \tag{29}$$

$$\frac{\partial}{\partial n}\left(\frac{\partial^2\phi}{\partial x\,\partial y} + \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right) + \frac{\partial}{\partial t}\left(\frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial x^2} - 4\frac{\partial^2\psi}{\partial x\,\partial y}\right) = 0, \quad \text{on } \Gamma_\epsilon, \tag{30}$$

$$4\frac{\partial}{\partial t}\left(\frac{\partial^2\phi}{\partial x\,\partial y} + \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right) - \frac{\partial}{\partial n}\left(\frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial x^2} - 4\frac{\partial^2\psi}{\partial x\,\partial y}\right) = 0, \quad \text{on } \Gamma_\epsilon. \tag{31}$$

Note that in deriving the natural boundary conditions (25)-(28) we have made use of the essential boundary conditions satisfied by the stream function. The essential boundary conditions for the stream function will always be assumed to hold for both the exact and the discrete problems. Conditions (25)-(31) are the natural boundary conditions for the Stokes problem. Since $\partial^2\phi/(\partial x\,\partial y) = 0$ along B^+C and DE, it follows that $\partial^3\phi/(\partial x^2\,\partial y) = 0$ there. Therefore the first condition in (28) becomes $\partial^3\phi/\partial y^3 = 0$ along B^+C and DE.

We proceed to establish the essential boundary conditions on the Stokesian Airy stress function. At entry we impose $\partial p/\partial x = -3$. From (2) we see that $\partial p/\partial y = 0$ at entry and hence $p(x, y) \to -3x - 2\alpha_{en}$ asymptotically as $x \to -\infty$ for some constant α_{en} which is at present unknown.

Eqs. (6) become, as $x \to -\infty$,

$$3x + 2\alpha_{en} = \frac{\partial^2\phi}{\partial y^2}, \qquad 3x + 2\alpha_{en} = \frac{\partial^2\phi}{\partial x^2}, \qquad -3y = -\frac{\partial^2\phi}{\partial x\,\partial y}, \tag{32}$$

which may be solved to give

$$\phi(x, y) = \tfrac{1}{2}x^3 + \tfrac{3}{2}xy^2 + \alpha_{en}(x^2 + y^2) + \beta_{en}x + \gamma_{en}y + \delta_{en}, \tag{33}$$

as $x \to -\infty$, where β_{en}, γ_{en} and δ_{en} are arbitrary constants.

Eqs. (6) may also be solved as $x \to \infty$ with $\lim_{x\to\infty} p(x, y)$ assigned some constant value $-2\alpha_{ex}$, say, and in this case we obtain

$$\phi(x, y) = \alpha_{ex}(x^2 + y^2) + \beta_{ex}x + \gamma_{ex}y + \delta_{ex}, \tag{34}$$

as $x \to \infty$, with β_{ex}, γ_{ex} and δ_{ex} arbitrary constants.

It will be clear that the constants $\beta_{en}, \gamma_{en}, \delta_{en}, \beta_{ex}, \gamma_{ex}$ and δ_{ex} are of no physical significance and they are neglected accordingly. Since the pressure field is unique up to the addition of an arbitrary constant, we are free to choose $\alpha_{ex} = 0$. So without loss of generality we write

$$\phi(x, y) = \begin{cases} \tfrac{1}{2}x(x^2 + 3y^2) + \alpha_{en}(x^2 + y^2), & \text{as } x \to -\infty, \\ 0, & \text{as } x \to \infty. \end{cases}$$

The shear-free condition $\partial^2\phi/(\partial x\,\partial y) = 0$ on B^+C and DE now becomes

$$\frac{\partial \phi}{\partial y} = 0, \quad \text{on } B^+C \text{ and } DE,$$

since $\partial\phi/\partial y = 0$ on CD.

Let V and W be two sets of admissible pairs of functions (η, ζ) which are four times continuously differentiable in Ω and which are such that for $(\psi, \phi) \in V$, ψ satisfies the essential boundary conditions (11)–(15) and for $(\eta, \zeta) \in W$, η satisfies the homogeneous equivalents of these boundary conditions. The minimization problem (20) then has the equivalent variational formulation. Find $(\psi, \phi) \in V$ such that

$$\int\int_\Omega [(L\phi - 2M\psi)(L\eta - 2M\zeta) + (M\phi + 2L\psi)(M\eta + 2L\zeta)]\,\mathrm{d}S = 0,$$

for all $(\eta, \zeta) \in W$. The corresponding discrete variational formulation can be written down in a similar manner by introducing the associated discrete spaces.

4. Regularity assumptions

Let us define the functions $\bar{\phi}$ and $\bar{\psi}$ as follows:

$$\bar{\phi}(x,y) = \begin{cases} \phi(x,y) - \frac{1}{2}x(x^2 + 3y^2) - \alpha_{\text{en}}(x^2 + y^2), & x \leqslant 0, \\ \phi(x,y), & x > 0, \end{cases} \tag{35}$$

$$\bar{\psi}(x,y) = \begin{cases} \psi(x,y) - \frac{1}{2}y(3 - y^2), & x \leqslant 0, \\ \psi(x,y) - y, & x > 0. \end{cases} \tag{36}$$

We assume that $\bar{\phi}$ and $\bar{\psi}$ satisfy the conditions

$$\frac{\partial^n \bar{\phi}}{\partial x^n} \to 0, \quad \text{as } x \to \pm\infty, \tag{37}$$

$$\frac{\partial^n \bar{\psi}}{\partial x^n} \to 0, \quad \text{as } x \to \pm\infty, \tag{38}$$

for $n = 1, 2, 3$. These conditions ensure that the entry and exit conditions (33) and (34) imply the natural conditions (25) and (26) on ϕ. Similarly, regularity in the sense of (37) and (38) guarantees that the entry and exit conditions (14) and (15) on ψ imply

$$\frac{\partial^n \psi}{\partial x^n} \to 0, \quad \text{as } x \to \pm\infty, \tag{39}$$

for $n = 1, 2, 3$. Conditions (37) and (38) may be shown to follow from the assumptions made in deriving the entry and exit profiles (14) and (15) for the stream function and (25) and (26) for the Airy stress function. Our numerical approximations to ϕ and ψ will be constructed to satisfy these conditions.

5. Uniqueness of solution

The least-squares formulation of the elliptic system (7), (8) has led to a system of fourth-order equations with natural boundary conditions. Before we show that this system has a unique solution, we establish the asymptotic forms of the dependent variables ϕ and ψ in a neighbourhood of the stick-slip point $(0,1)$. A knowledge of these asymptotic forms will be crucial in extending the preceding analysis to Ω_h. We also show that for a carefully chosen value of α_{en}, the boundary conditions ensure that the solution of the embedded system which also solves the second-order elliptic system (7), (8) is retrieved.

5.1. Asymptotic form of ψ

In a manner analogous to that of [17], Richardson [20] calculated the asymptotic form of the stream function ψ in a sufficiently small neighbourhood of the boundary singularity for Stokes flow. Local polar coordinates (r,θ) centred at the boundary singularity $(0,1)$ are taken. In a vicinity of the singularity, a behaviour of the form $\psi \sim 1+r^{\lambda+1}f_\lambda(\theta)$ may be expected with $\lambda > 0$ to give finite integrable stresses at $r=0$. Since the governing equation for ψ is the biharmonic equation, the form for $f_\lambda(\theta)$ is readily determined. The arbitrary constants are determined from the boundary conditions

$$\psi = 1, \quad \text{on } \theta = 0, \pi, \qquad \frac{\partial\psi}{\partial y} = 0, \quad \text{on } \theta = 0, \qquad \frac{\partial^2\psi}{\partial y^2} = 0, \quad \text{on } \theta = \pi,$$

whereupon an eigenvalue problem for λ arises. The two possible sets of solutions are

(a) $1 + A_n r^{(2n+1)/2}\sin\{\frac{1}{2}(2n-1)\theta\}\sin\theta$,

(b) $1 + B_n r^{n+2}\{(n+1)\cos\{(n+1)\theta\}\sin\theta - \sin\{(n+1)\theta\}\cos\theta\}$,

for all positive integral values of n where A_n and B_n are constants. An expansion for ψ about $r=0$ will involve a sum of the trigonometric terms in (a) and (b) in addition to the constant term. The dominant terms in this expansion are

$$1 + A_1 r^{3/2}\sin(\tfrac{1}{2}\theta)\sin\theta. \tag{40}$$

5.2. Asymptotic form of ϕ

We solve the biharmonic equation for ϕ subject to the boundary conditions

$$\frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial x^2} = 0, \quad \frac{\partial^3\phi}{\partial y^3} + 3\frac{\partial^3\phi}{\partial x^2\,\partial y} = 0, \qquad \text{on } \theta = 0,$$

$$\frac{\partial^3\phi}{\partial y^3} = 0, \quad \frac{\partial\phi}{\partial y} = 0, \qquad \text{on } \theta = \pi.$$

The dominant term in the asymptotic series representing ϕ is written as $r^\mu g(\theta)$. The boundary conditions when expressed as a function of g are

$$(2\mu - \mu^2)g(0) + g''(0) = 0, \qquad g'''(0) + (3\mu^2 - 6\mu + 4)g'(0) = 0,$$

$$(3\mu - 2)g'(\pi) + g'''(\pi) = 0, \qquad g'(\pi) = 0.$$

The smallest eigenvalue μ of the fourth-order ordinary differential equation for g, which allows the stresses to be integrable, is found to be $\mu = \frac{3}{2}$ and the behaviour of ϕ as $r \to 0$ is therefore dominated by

$$\phi \sim r^{3/2} \sin\theta \cos(\tfrac{1}{2}\theta). \tag{41}$$

5.3. Calculation of α_{en}

The value of α_{en} is chosen in such a way that the pressure p is continuous in a weak sense across the line $x = 0$, i.e., we require

$$\int_0^1 p(0-, y)\,dy = \int_0^1 p(0+, y)\,dy. \tag{42}$$

In order to determine α_{en}, we first observe that (2) implies

$$-\frac{\partial p}{\partial x} + \frac{\partial^3 \psi}{\partial x^2\,\partial y} + \frac{\partial^3 \psi}{\partial y^3} = 0,$$

and hence that

$$p(-\epsilon, y) - p(x, y) = \int_x^{-\epsilon} \frac{\partial^3 \psi}{\partial \zeta^2\,\partial y}(\zeta, y)\,d\zeta + \int_x^{-\epsilon} \frac{\partial^3 \psi}{\partial y^3}(\zeta, y)\,d\zeta,$$

for $-\infty < x \leqslant -\epsilon$. We integrate with respect to y from 0 to 1 to obtain

$$\int_0^1 p(-\epsilon, y)\,dy - \int_0^1 p(x, y)\,dy = \int_0^1 \int_x^{-\epsilon} \frac{\partial^3 \psi}{\partial \zeta^2\,\partial y}(\zeta, y)\,d\zeta\,dy + \int_0^1 \int_x^{-\epsilon} \frac{\partial^3 \psi}{\partial y^3}(\zeta, y)\,d\zeta\,dy. \tag{43}$$

If we assume that $\psi \in L^1(\mathbb{R}^2)$, we can interchange the order of integration on the right-hand side. Therefore,

$$\begin{aligned}\int_0^1 p(-\epsilon, y)\,dy - \int_0^1 p(x, y)\,dy &= \int_x^{-\epsilon} \left[\frac{\partial^2 \psi}{\partial \zeta^2}\right]_{y=0}^{y=1} d\zeta + \int_x^{-\epsilon} \left[\frac{\partial^2 \psi}{\partial y^2}\right]_{y=0}^{y=1} d\zeta \\ &= \int_x^{-\epsilon} \frac{\partial^2 \psi}{\partial y^2}(\zeta, 1)\,d\zeta. \end{aligned} \tag{44}$$

We now let $x \to -\infty$ and recall that $\bar{\psi}(x, y) = \psi(x, y) - \frac{1}{2}y(3 - y^2)$ to conclude that

$$\int_0^1 p(-\epsilon, y)\,dy + 2\alpha_{en} = \int_{-\infty}^{-\epsilon} \frac{\partial^2 \bar{\psi}}{\partial y^2}(\zeta, 1)\,d\zeta.$$

Finally, we let $\epsilon \to 0$. There is no problem in doing this since $\partial^2\bar{\psi}/\partial y^2 = O(r^{-1/2})$ as $r \to 0$. Hence,

$$\int_0^1 p(0-, y)\,dy + 2\alpha_{en} = -\int_{AB} \frac{\partial^2 \bar{\psi}}{\partial y^2}\,ds. \tag{45}$$

Following a similar argument to the above, one also obtains

$$\int_0^1 p(0+, y)\,dy = 0, \tag{46}$$

so that weak continuity of the pressure across $x = 0$ results in the condition

$$\alpha_{\text{en}} = -\frac{1}{2}\int_{AB} \frac{\partial^2 \bar{\psi}}{\partial y^2}\,\mathrm{d}s. \tag{47}$$

In the remainder of this paper we assume that α_{en} is defined by this expression.

6. Equivalence and uniqueness results

Theorem 6.1. *If there exists a solution of the following embedded fourth-order problem, then it solves the second-order problem* (7), (8):

$$\phi = \tfrac{1}{2}x(x^2 + 3y^2) + \alpha_{\text{en}}(x^2 + y^2), \quad \psi = \tfrac{1}{2}y(3 - y^2), \qquad \textit{on } AE, \tag{48}$$

$$\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial x^2} = 0, \quad \frac{\partial^3 \phi}{\partial y^3} + 3\frac{\partial^3 \phi}{\partial x^2\,\partial y} = 0, \quad \psi = 1, \quad \frac{\partial \psi}{\partial y} = 0, \qquad \textit{on } AB, \tag{49}$$

$$\frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial^3 \phi}{\partial y^3} = 0, \quad \psi = 1, \quad \frac{\partial^2 \psi}{\partial y^2} = 0, \qquad \textit{on } BC, \tag{50}$$

$$\frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial^3 \phi}{\partial y^3} = 0, \quad \psi = 0, \quad \frac{\partial^2 \psi}{\partial y^2} = 0, \qquad \textit{on } DE, \tag{51}$$

$$\phi = 0, \quad \psi = y, \qquad \textit{on } CD, \tag{52}$$

$$\Delta^2 \phi = 0, \quad \Delta^2 \psi = 0, \qquad \textit{in } \Omega_h. \tag{53}$$

Proof. Let us set $q = u = \phi$, $r = \psi$ and $v = \psi - y$ in (16). Then from the boundary conditions (48)–(52) and the regularity conditions (37), (38), all the terms on the right-hand side of (16) are seen to vanish immediately except for the integrals calculated around Γ_ϵ and the integral

$$\int_{\Gamma \backslash \Gamma_\epsilon} \frac{\partial v}{\partial n}(M\phi + 2L\psi)\,\mathrm{d}s. \tag{54}$$

The asymptotic form of ϕ and ψ guarantees that the contribution from line integrals evaluated around Γ_ϵ all vanish as $\epsilon \to 0$, since for all but the last line integral on the right-hand side of (16) the integrands are O(1) as $\epsilon \to 0$. The last line integral is

$$I(\epsilon) = 2\int_{\Gamma_\epsilon} (\psi - y)\left\{\frac{\partial}{\partial n}(M\phi + 2L\psi) + 2\frac{\partial}{\partial t}(L\phi - 2M\psi)\right\}\mathrm{d}s. \tag{55}$$

Since

$$y = 1 - \epsilon \sin\theta, \qquad \text{on } \Gamma_\epsilon,$$

we have, as $\epsilon \to 0$,

$$\psi - y \sim \epsilon^{3/2} \sin\theta \sin(\tfrac{1}{2}\theta) + \epsilon \sin\theta.$$

Also, since the terms in braces in (55) are $O(\epsilon^{-3/2})$, the dominant term in the integrand of $I(\epsilon)$ is $O(\epsilon^{-1/2})$. Noting that $ds = \epsilon\, d\theta$, we see that

$$\lim_{\epsilon \to 0} I(\epsilon) = 0,$$

as required.

The boundary and regularity conditions imply $M\phi + 2L\psi$ is zero everywhere on the boundary except along AB^-, so that we deduce

$$\int\int_{\Omega} [(L\phi - 2M\psi)^2 + (M\phi + 2L\psi)^2]\, dS = 2\int_{AB^-} (M\phi + 2L\psi)\, ds. \tag{56}$$

We may let $\epsilon \to 0$, as before, to equate the double integral evaluated over the region Ω_h with that of the right-hand side integral evaluated over AB, i.e., as $\epsilon \to 0$,

$$\int\int_{\Omega_h} [(L\phi - 2M\psi)^2 + (M\phi + 2L\psi)^2]\, dS = 2\int_{AB} (M\phi + 2L\psi)\, ds. \tag{57}$$

Now the right-hand side of (57) is

$$4\int_{AB} \left(\frac{\partial^2 \phi}{\partial x\, \partial y} + \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} \right) dx,$$

which, since $\partial^2\psi/\partial x^2 = 0$ along AB, becomes

$$4\left[\frac{\partial \phi}{\partial y}\right]_B^A + 4\int_{AB} \frac{\partial^2 \psi}{\partial y^2}\, dx. \tag{58}$$

The right-hand side of (57) would be the same as if the integral were to be evaluated over AC instead of AB, which from (59) implies that

$$\frac{\partial \phi}{\partial y}(0, 1) = \frac{\partial \phi}{\partial y}(\infty, 1). \tag{59}$$

Eqs. (52) and (59) together yield the result

$$\frac{\partial \phi}{\partial y}(0, 1) = 0,$$

so that, using the knowledge of ϕ at entry, the right-hand side of (57) may be shown to be

$$8\alpha_{en} + 4\int_{AB} \frac{\partial^2 \bar{\psi}}{\partial y^2}\, ds,$$

which is zero from (47).

Thus, since the integrand on the left-hand side of (57) is strictly nonnegative, it must be zero and therefore,

$$L\phi - 2M\psi = 0, \qquad M\phi + 2L\psi = 0,$$

everywhere in Ω_h. $\square$

Theorem 6.2. *If there exists a solution of the embedded system* (48)–(53) *in the stick-slip geometry, then it is unique.*

Proof. Let ϕ_1, ψ_1 and ϕ_2, ψ_2 be two solutions of the embedded system with $\phi_1 \neq \phi_2$ and $\psi_1 \neq \psi_2$. Let $\Phi = \phi_1 - \phi_2$ and $\Psi = \psi_1 - \psi_2$. Then Φ and Ψ satisfy the following:

$$\Delta^2\Phi = 0, \quad \Delta^2\Psi = 0, \qquad \text{in } \Omega_h, \tag{60}$$

$$\Phi = 0, \quad \Psi = 0, \qquad \text{on } AE \text{ and } CD, \tag{61}$$

$$\frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Phi}{\partial x^2} = 0, \quad \frac{\partial^3\Phi}{\partial y^3} + 3\frac{\partial^3\Phi}{\partial x^2\,\partial y} = 0, \qquad \text{on } AB, \tag{62}$$

$$\Psi = 0, \quad \frac{\partial\Psi}{\partial y} = 0, \qquad \text{on } AB, \tag{63}$$

$$\frac{\partial\Phi}{\partial y} = 0, \quad \frac{\partial^3\Phi}{\partial y^3} = 0, \quad \Psi = 0, \quad \frac{\partial^2\Psi}{\partial y^2} = 0, \qquad \text{on } BC \text{ and } DE. \tag{64}$$

Setting $q = u = 0$ and $r = v = \Psi$ in (16), we obtain

$$4\int\int_\Omega \{(M\Psi)^2 + (L\Psi)^2\}\,\mathrm{d}S = -4\int_\Gamma \frac{\partial\Psi}{\partial n} L\Psi\,\mathrm{d}s + 4\int_\Gamma \Psi\left\{\frac{\partial}{\partial n}(L\Psi) - 2\frac{\partial}{\partial t}(M\Psi)\right\}\mathrm{d}s.$$

From (61)–(64) and the regularity conditions (37), (38) we observe that $\Psi = 0$ and either $\partial\Psi/\partial n = 0$ or $L\Psi = 0$ on $\Gamma \setminus \Gamma_\epsilon$. Furthermore, on $\Gamma_\epsilon, \Psi = O(\epsilon^{3/2})$, so that $(\partial\Psi/\partial n)(L\Psi)$, $\Psi(\partial/\partial n)(L\Psi)$ and $\Psi(\partial/\partial t)(M\Psi)$ are all $O(1)$. Since on $\Gamma_\epsilon, \mathrm{d}s = \epsilon\,\mathrm{d}\theta$, then, as $\epsilon \to 0$, we obtain

$$\int\int_{\Omega_h} \{(M\Psi)^2 + (L\Psi)^2\}\,\mathrm{d}S = 0. \tag{65}$$

Since each of the terms of the integrand are positive, it follows that both must be identically zero in Ω_h. Thus,

$$M\Psi = 0, \quad L\Psi = 0, \qquad \text{in } \Omega_h.$$

It follows that Ψ must take the form

$$\Psi(x, y) = a(x^2 + y^2) + bx + cy + d, \tag{66}$$

where a, b, c, d are constants. The conditions (61)–(64) on Ψ imply that all these constants are zero and therefore $\Psi = 0$ everywhere in Ω_h.

A similar argument follows for Φ on setting $q = u = \Phi$, $r = v = 0$ in (16). Again we find that Φ must be of the form

$$\Phi(x, y) = a(x^2 + y^2) + bx + cy + d. \tag{67}$$

As before, the conditions (61)–(64) on Φ force the constants to be zero, so that $\Phi = 0$ everywhere in Ω_h.

Thus we have shown that if a solution to the embedded problem (48)–(53) exists, then it is unique. □

The discussion thus far has assumed a flow domain which is truncated in some sense. This has been necessary to develop the boundary condition theory. For the purpose of the numerical experiments, however, we shall assume an infinite channel, so that we do not need to worry about the determination of a suitable value of h.

7. Domain decomposition method

The flow domain is decomposed into two semi-infinite rectangles having $x = 0$ as their interface, as shown in Fig. 1. Global approximations to the solution are used in each subdomain and the solutions are patched across the interface [12,13] using C^3 continuity conditions. Let R_{I} and R_{II} denote the two subdomains. Then we may map R_{I} onto the finite rectangle $M_{\mathrm{I}} = \{(s, y)\colon\ s \in [0, 1], y \in [0, 1]\}$ by defining the algebraic map

$$s = -\frac{x}{L_{\mathrm{I}} - x},$$

where $L_{\mathrm{I}} > 0$ is some mapping parameter. Similarly R_{II} may be mapped onto $M_{\mathrm{II}} = \{(t, y)\colon\ t \in [0, 1], y \in [0, 1]\}$ using the map

$$t = \frac{x}{L_{\mathrm{II}} + x},$$

where $L_{\mathrm{II}} > 0$ is another mapping parameter.

In the transformed regions we approximate ψ by $\psi^{\mathrm{I}}(x, y)$ and $\psi^{\mathrm{II}}(x, y)$ where

$$\psi^{\mathrm{I}}(x, y) = \tfrac{1}{2}y(3 - y^2) + \sum_{m=4}^{M} \sum_{n=0}^{L} a_{mn}^{\mathrm{I}} R_m(y) T_n^*(s) \tag{68}$$

and

$$\psi^{\mathrm{II}}(x, y) = y + \sum_{m=4}^{M} \sum_{n=0}^{L} a_{mn}^{\mathrm{II}} S_m(y) T_n^*(t), \tag{69}$$

where T_n^* is the shifted Chebyshev polynomial of degree n defined over the interval $[0, 1]$, defined by

$$T_n^*(z) = T_n(2z - 1) = \cos(n \cos^{-1}(2z - 1)), \quad z \in [0, 1].$$

We note that

$$T_n^*(t) = T_n\left(\frac{x - L_{\mathrm{II}}}{x + L_{\mathrm{II}}}\right) = LT_n(x),$$

where LT_n is the rational Chebyshev polynomial of degree n defined over $[0, \infty)$ and described in [1].

The polynomial R_m is defined by

$$R_m(y) = T_m^*(y) + \alpha_m T_3^*(y) + \beta_m T_2^*(y) + \gamma_m T_1^*(y) + \delta_m, \quad m \geqslant 4,$$

and the polynomial S_m by

$$S_m(y) = T_m^*(y) + \kappa_m T_3^*(y) + \lambda_m T_2^*(y) + \mu_m T_1^*(y) + \nu_m, \quad m \geqslant 4,$$

where the coefficients $\alpha_m, \beta_m, \gamma_m, \delta_m, \kappa_m, \lambda_m, \mu_m$ and ν_m are chosen so that (68) satisfies the boundary conditions (11) and (13) and so that (69) satisfies the boundary conditions (12) and (13). These boundary conditions require that

$$\alpha_m = \tfrac{1}{128}(T_m^{*\prime\prime}(0) - 2(T_m^{*\prime}(1) + T_m^*(0) - T_m^*(1))), \qquad \beta_m = \tfrac{1}{16}(96\alpha_m - T_m^{*\prime\prime}(0)),$$

$$\gamma_m = \tfrac{1}{2}(T_m^*(0) - T_m^*(1)) - \alpha_m, \qquad \delta_m = -\tfrac{1}{2}(T_m^*(0) + T_m^*(1)) - \beta_m,$$

and that

$$\kappa_m = \tfrac{1}{192}(T_m^{*\prime\prime}(0) - T_m^{*\prime\prime}(1)), \qquad \lambda_m = -\tfrac{1}{32}(T_m^{*\prime\prime}(0) + T_m^{*\prime\prime}(1)),$$

$$\mu_m = \tfrac{1}{2}(T_m^*(0) - T_m^*(1)) - \kappa_m, \qquad \nu_m = -\tfrac{1}{2}(T_m^*(0) + T_m^*(1)) - \lambda_m.$$

In a similar fashion we approximate the Airy stress function $\phi(x,y)$ by $\phi^{\mathrm{I}}(x,y)$ in region M_{I} and by $\phi^{\mathrm{II}}(x,y)$ in region M_{II} where

$$\phi^{\mathrm{I}}(x,y) = \tfrac{1}{2}x^3 + \tfrac{3}{2}xy^2 + \alpha_{\mathrm{en}}(x^2+y^2) + \sum_{m=0}^{M}\sum_{n=0}^{L} b_{mn}^{\mathrm{I}} T_m^*(y) T_n^*(s) \tag{70}$$

and

$$\phi^{\mathrm{II}}(x,y) = \sum_{m=0}^{M}\sum_{n=0}^{N} b_{mn}^{\mathrm{II}} T_m^*(y) T_n^*(t). \tag{71}$$

In the spectral approximations (68)–(71) the quantities $\{a_{mn}^{\mathrm{I}}\}$, $\{a_{mn}^{\mathrm{II}}\}$, $\{b_{mn}^{\mathrm{I}}\}$ and $\{b_{mn}^{\mathrm{II}}\}$ are expansion coefficients to be determined. These unknown coefficients are found by satisfying the differential equations (24), the boundary conditions on ϕ and ψ and the interface continuity conditions at appropriately chosen collocation points.

For the choice of the collocation points in each coordinate direction we follow [18] and use the extrema of the shifted Chebyshev polynomial of highest degree in that variable used in the expansion representation. These are simple adaptations of the Gauss–Lobatto points, described, for example, in [4]. The discretization is chosen to be nonconforming due to the nature of the singularity at the stick-slip point. The use of conforming spectral discretizations would have the effect of smoothing out the singularity, see [19] for details of this collocation scheme. The choice of mapping parameters controls the distribution of collocation points in the x-direction.

The basis functions which we use in the approximations (68)–(71) ensure that the regularity conditions (37) and (38) are satisfied automatically. In order to determine the constant α_{en} in (70), we first solve the stream function problem and then calculate α_{en} from relation (47) using Simpson's rule to evaluate the integral numerically.

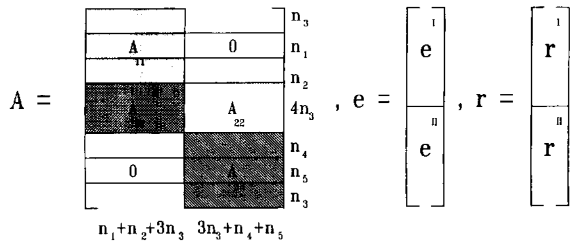

Fig. 2. The algebraic structure of the system of equations for the unknowns in the representation of the Airy stress function.

8. Method of solution

The systems of equations for the expansion coefficients in the approximations to ϕ and ψ give rise to large coefficient matrices possessing significant submatrices of zero elements. In this section we describe a technique suited to the underlying structure of the matrix which provides savings in the computation time and storage required over the standard direct solution method. The solution procedure for the stream function problem follows by analogy with that of the Airy stress.

The global system for the expansion coefficients in the approximation to the Airy stress function ϕ is

$$A\boldsymbol{b} = \boldsymbol{r}, \tag{72}$$

where A, $\boldsymbol{b}$ and $\boldsymbol{r}$ are as shown in Fig. 2.

System (72) may be solved by using an application of a code designed for the solution of almost block-diagonal systems (see [2]). In a recent survey of direct methods for solving systems of equations arising from spectral domain decomposition methods, Karageorghis and Phillips [14] found this solver to be superior to methods of direct inversion of the full system, capacitance matrix techniques [3] or coefficient-splitting strategies with respect to cost, stability and storage. The same authors used the solver for the solution of the flow of a Newtonian liquid through a planar contraction [15].

The code for the solver is based on an algorithm described in [6,23] and is intended to solve matrix systems of the form shown in Fig. 3. These "almost block-diagonal" systems comprise rectangular blocks along the matrix diagonal in such a way that no three successive blocks have columns in

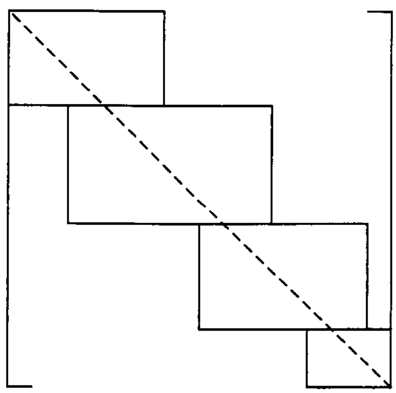

Fig. 3. A matrix in almost block-diagonal form.

common. The collocation matrix A in (72) may be seen to be in almost block-diagonal form with three nonzero blocks, namely

$$(A_{11}), \qquad (A_{21}|A_{22}), \qquad (A_{32}). \tag{73}$$

It is not possible to use the form (73) for the solver since there is too much overlap between blocks 1 and 2, and 2 and 3. Performing the transpose of A, however, gives the required almost block-diagonal form, now possessing two blocks

$$(A_{11}^{\mathrm{t}}|A_{21}^{\mathrm{t}}) \quad \text{and} \quad (A_{22}^{\mathrm{t}}|A_{32}^{\mathrm{t}}). \tag{74}$$

The transposed matrix A^{t} is decomposed using the NAG [24] routine F01LHF, the NAG routine F04LHF then being used to solve the system

$$(\bar{A}^{\mathrm{t}})^{\mathrm{t}}\boldsymbol{b} = \boldsymbol{r}, \tag{75}$$

where $\bar{A}^{\mathrm{t}}$ is the decomposed form of A^{t}.

9. Treatment of the stick-slip singularity

The accuracy of the numerical solution may be improved near the stick-slip point using a small number of terms in a singular expansion. The spectral approximation that we have obtained for the stream function is post-processed in order to obtain the coefficients in the singular expansion. Since in most elliptic problems the pollutive effect of boundary singularities does not penetrate far into the interior of the region [7], the approximations (68) and (69) converge quite rapidly away from the singularity. However, near the singularity these expansions converge slowly, and one may require many terms to predict accurately the correct singular behaviour there.

Define a neighbourhood of the singularity by

$$S = \{(r,\theta):\ 0 \leqslant r \leqslant R, 0 \leqslant \theta \leqslant \pi\},$$

where $r^2 = x^2 + (y-1)^2$ and θ is the polar angle. Then, following the analysis of [9], we can show that the solution of the biharmonic problem for the stream function has the form

$$\psi(r,\theta) = 1 + \sum_{k=1}^{\infty} a_k r^{1+\sqrt{\lambda_k}} u_k(\theta), \tag{76}$$

where

$$u_k(\theta) = \begin{cases} \cos(\sqrt{\lambda_k}+1)\theta - \cos(\sqrt{\lambda_k}-1)\theta, & \text{if } k \text{ is odd}, \\ (\sqrt{\lambda_k}-1)\sin(\sqrt{\lambda_k}+1)\theta - (\sqrt{\lambda_k}+1)\sin(\sqrt{\lambda_k}-1)\theta, & \text{if } k \text{ is even}, \end{cases} \tag{77}$$

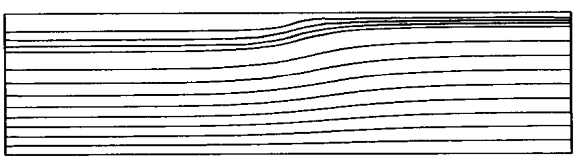

Fig. 4. Contours of the stream function plotted in $-2 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 1$, for MESH 2, for contour heights $0.1, 0.2, \ldots, 0.9, 0.925, 0.95, 0.975$.

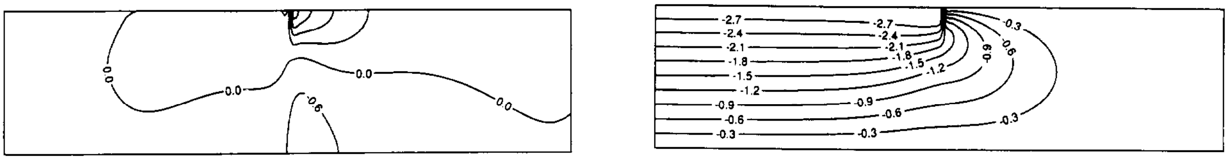

Fig. 5. Contours of (left) $\frac{1}{2}L\phi$, (right) $-\frac{1}{2}M\phi$, plotted in $-2 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 1$, for MESH 1.

and $\sqrt{\lambda_k} = \frac{1}{2}k$. Note that we can set $a_2 = 0$ since $u_s(\theta) \equiv 0$. The coefficients a_k are found using a post-processing technique described in [13]. In practice, the upper limit in (76) is replaced by a finite integer N, say. The solution of the truncated singular expansion is then valid within S. Outside S the solution of the biharmonic equation is given by either (68) or (69), depending on whether x is negative or positive, respectively.

We are unable to follow a similar analysis for the corresponding Airy stress problem because of the nonexistence of a biorthogonal set of adjoint eigenfunctions.

10. Numerical results

In this section we present the results of numerical calculations performed for the Stokesian planar stick-slip problem. Qualitative comparisons are made in the case of the streamlines and the pressure field for the problem with those predicted in [8,20]. Richardson [20] solved the planar stick-slip problem using the Wiener-Hopf technique and Georgiou et al. [8] used singular finite-element methods for the same problem. Two different collocation grids were used in the course of the numerical calculations. The importance of comparing results obtained with different grids lies in the necessity of showing that the numerical approximations are not heavily mesh-dependent. The three grids used were

(1) MESH 1: $L = M = N = 10$,
(2) MESH 2: $L = M = N = 15$,
(3) MESH 3: $L = M = N = 20$.

We define the mapping parameters to be the same in absolute value, i.e., $|L_{\mathrm{I}}| = |L_{\mathrm{II}}|$. In Fig. 4 we present a plot of the streamlines obtained with MESH 2 and observe a favourable comparison with the results of [8]. In Fig. 5 we plot $\frac{1}{2}L\phi$ and $-\frac{1}{2}M\phi$ for MESH 1. Figs. 6 and 7 show the pressure field for both meshes. The pressure field in Fig. 6 agrees closely with that obtained in [8], whereas in Fig. 7 the centreline pressure field is equivalent to those of [8,20]. The centreline pressure varies linearly with axial distance inside the confined channel and goes to zero smoothly in the jet. Oscillations in the centreline pressure with increasing numbers of degrees of freedom were alleviated by increasing the absolute value of the mapping parameters L_{I} and L_{II}. This effectively reduces the spectral condition number of the matrix systems and removes oscillations by sweeping the collocation

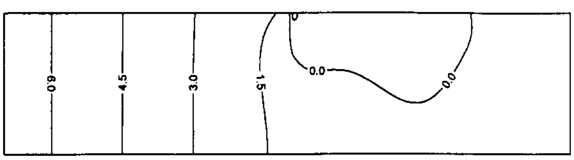

Fig. 6. Contours of the pressure in $-2 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 1$, for MESH 1.

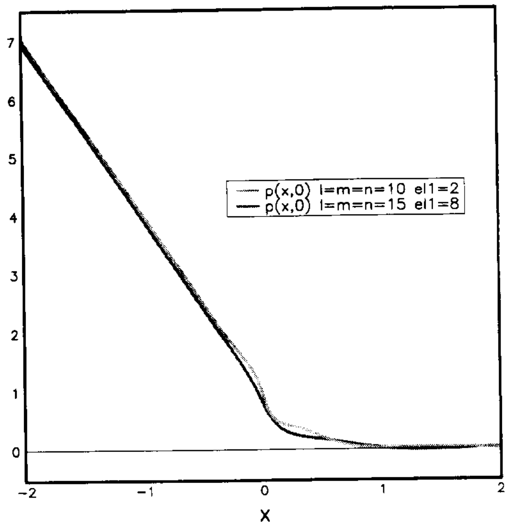

Fig. 7. Pressure along the centreline $x = 0$ for (a) MESH 1, $L_{II} = 2$, (b) MESH 2, $L_{II} = 8$.

points away from the interface and reducing the density of points there.

In Table 1 we present the calculated value of α_{en} for different numbers of degrees of freedom. The length of the interval over which the integral in (47) was evaluated was fixed at 1.5 and the number of Simpson subdivisions set at 1000. Outside this interval of integration the integrand is small, since we are far enough downstream for fully developed conditions to have been reached. The converged value appears to be approximately −0.4695 and agrees with what can be deduced from the centreline plots of [8,20].

The computation of the results shown in Tables 2–4 was performed on a mesh with $M = L = N = 25$. In Table 2 we show the dependence of the coefficients in the singular expansion on the mapping parameter L_I when $R = 0.75$. This demonstrates the convergence of these coefficients on a fixed mesh with respect to the mapping parameter.

In Table 3 we show how the leading coefficients in the singular expansion (76) for the stream function depend on the order of the spectral approximations (68) and (69) and R, the radius of the sector in which the singular expansion is valid. In this table the values of the coefficients can be seen to be converging to some finite value as L is increased. We expect the coefficients to become independent of R for values of R sufficiently large and there is evidence of this in the table. In Table 4 we compare our numerically calculated coefficients with the analytical solution for a_1 and previous work. Kelmanson [16] used a singular boundary integral method and Karageorghis [11]

Table 1
Calculated values of α_{en} for different orders of approximation N

N	α_{en}
10	−0.7798
15	−0.4596
20	−0.5813
25	−0.4732
30	−0.4695

Table 2
Dependence of the coefficients in the singular expansion on the mapping parameter L_I

L_I	a_1	a_3	a_5
1	0.68937	0.26310	0.03152
2	0.69035	0.26404	0.03069
3	0.69037	0.26408	0.03058

Table 3
Dependence of the coefficients in the singular expansion (76) on M and R

M	R	a_1	a_3	a_5
10	0.25	0.72921	0.21999	−0.00953
10	0.50	0.72037	0.28540	0.00598
10	0.75	0.70865	0.27796	0.01959
20	0.25	0.68938	0.25900	0.03411
20	0.50	0.69021	0.26351	0.03108
20	0.75	0.69101	0.26469	0.03011
25	0.25	0.68898	0.26024	0.03478
25	0.50	0.68970	0.26321	0.03144
25	0.75	0.69035	0.26404	0.03069

used a modified version of the method of fundamental solutions which incorporates the singular behaviour of the problem. The agreement with the analytical solution is satisfactory. The second and third singular coefficients agree to three decimal places to the ones calculated in [16]. The contour plots of the stream function are the same as those shown in Fig. 5 and therefore they are not given here.

11. Conclusions

The equations describing conservation of mass and momentum are satisfied identically by introducing a stream function and Airy stress function as dependent variables into the governing equations for Stokes flow. This process results in a coupled second-order elliptic system. A pair of biharmonic equations for the stream function and Airy stress function is derived from a least-squares formulation of the second-order system. The two variables are coupled through the natural boundary conditions, in general. The technique is applied to the stick-slip problem. The asymptotic form of the singularity at the stick-slip point is derived for both the stream function and Airy stress function. The natural boundary conditions are shown to be sufficient to recover the solution of the second-order system, provided a solution exists. Some numerical results which are obtained using spectral domain decomposition techniques are presented. The coefficients in a singular expansion of the stream function around the stick-slip are computed and compared with those obtained by other workers.

The process we have described may be applied to the Navier–Stokes equation by appropriately defining an Airy stress function so that the equations of motion are satisfied exactly. The extension to problems in non-Newtonian fluid mechanics is more complex due to the nonlinear nature of the stress-strain relationship.

Table 4
Comparison of the calculated coefficients in the singular expansion (76) with the analytical solution and previous work

	a_1	a_3	a_4	a_5
Analytical	0.69099	—	—	—
Kelmanson [16]	0.69108	0.26435	−0.07990	0.04962
Georgiou et al. [8]	0.69173	0.27168	—	0.05013
Karageorghis [11]	0.69098	0.27481	−0.04398	−0.02210
Present work	0.69035	0.26404	−0.08051	0.03069

It is essential to design an effective preconditioner for the algebraic systems for the unknowns in the Airy stress function representation. This is because the condition numbers of the algebraic systems arising from a spectral discretization of fourth-order differential equations are high. This is an area for future research.

Acknowledgements

The first author acknowledges a European Science Exchange Programme Fellowship awarded by the Royal Society in conjunction with the Turkish Scientific and Technical Foundation (TUBITAK) held for part of the research period. The first author also acknowledges the support of the Boğaziçi Research Foundation for providing some of the computational resources used in this work.

References

[1] J.P. Boyd, Orthogonal rational functions on a semi-infinite interval, *J. Comput. Phys.* **70** (1987) 63–88.
[2] R.W. Brankin and I. Gladwell, Codes for almost block diagonal systems, *Comput. Math. Appl.* **19** (1990) 1–6.
[3] B.L. Buzbee, F.W. Dorr, J.A. George and G.H. Golub, The direct solution of the discrete Poisson equation on irregular regions, *SIAM J. Numer. Anal.* **8** (1971) 722–736.
[4] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, *Spectral Methods in Fluid Dynamics* (Springer, Berlin, 1988).
[5] C.J. Coleman, A contour integral formulation of plane creeping Newtonian flow, *Quart. J. Mech. Appl. Math.* **34** (1981) 453–464.
[6] J.C. Diaz, G. Fairweather and P. Keast, FORTRAN packages for solving certain almost block diagonal linear systems by modified alternate row and column eliminations, *ACM Trans. Math. Software* **9** (1983) 358–375.
[7] L. Fox, Finite differences and singularities in elliptic problems, in: I. Gladwell and R. Wait, Eds., *A Survey of Numerical Methods for Partial Differential Equations* (Clarendon Press, Oxford, 1979) 42–69.
[8] G.C. Georgiou, L.G. Olson, W.W. Schultz and S. Sagan, A singular finite element for Stokes flow: the stick-slip problem, *Internat. J. Numer. Methods Fluids* **9** (1989) 1353–1367.
[9] G. Geymonat and P. Grisvard, Eigenfunction expansions for non-self-adjoint operators and separation of variables, in: P. Grisvard, W. Wendland and J.R. Whiteman, Eds., *Singularities and Constructive Methods for their Treatment* (Springer, Berlin, 1985) 123–136.
[10] F.B. Hildebrand, *Methods of Applied Mathematics* (Prentice-Hall, Englewood Cliffs, NJ, 1965).
[11] A. Karageorghis, Modified methods of fundamental solutions for harmonic and biharmonic problems with boundary singularities, *Numer. Methods Partial Differential Equations* **8** (1992) 1–19.
[12] A. Karageorghis and T.N. Phillips, Spectral collocation methods for Stokes flow in contraction geometries and unbounded domains, *J. Comput. Phys.* **80** (1989) 314–330.
[13] A. Karageorghis and T.N. Phillips, Chebyshev spectral collocation methods for laminar flow through a channel contraction, *J. Comput. Phys.* **84** (1989) 114–133.
[14] A. Karageorghis and T.N. Phillips, On efficient direct methods for conforming spectral domain decomposition techniques, *J. Comput. Appl. Math.* **33** (2) (1990) 141–155.
[15] A. Karageorghis and T.N. Phillips, Conforming Chebyshev spectral collocation methods for the solution of laminar flow in a constricted channel, *IMA J. Numer. Anal.* **11** (1991) 33–54.
[16] M.A. Kelmanson, An integral equation method for the solution of singular slow flow problems, *J. Comput. Phys.* **51** (1983) 139–158.
[17] H.K. Moffatt, Viscous and resistive eddies near a sharp corner, *J. Fluid Mech.* **18** (1964) 1–18.
[18] S.A. Orszag, Spectral methods for problems in complex geometries, *J. Comput. Phys.* **37** (1980) 70–92.
[19] R.G. Owens and T.N. Phillips, A spectral domain decomposition method for the planar non-Newtonian stick-slip problem, *J. Non-Newtonian Fluid Mech.* **41** (1991) 43–79.

[20] S. Richardson, A "stick-slip" problem related to the motion of a free jet at low Reynolds numbers, *Proc. Cambridge Philos. Soc.* **67** (1970) 477–489.

[21] L.D. Sturges, A theoretical study of extrudate swell, Ph.D. Thesis, Univ. Minnesota, 1977.

[22] S.A. Trogdon and D.D. Joseph, The stick-slip problem for a round jet. I. Large surface tension, *Rheol. Acta* **19** (1980) 404–420.

[23] J.M. Varah, Alternate row and column elimination for solving certain linear systems, *SIAM J. Numer. Anal.* **13** (1976) 71–75.

[24] Numerical Algorithms Group Mark 15, NAG (UK) Ltd, Banbury Road, Oxford, UK, 1993.